



Booker, A. R., Krishnamurthy, M., & Lee, M. (2020). Test vectors for Rankin–Selberg L-functions. *Journal of Number Theory*, 209, 37-48. <https://doi.org/10.1016/j.jnt.2019.08.010>

Peer reviewed version

License (if available):
CC BY-NC-ND

Link to published version (if available):
[10.1016/j.jnt.2019.08.010](https://doi.org/10.1016/j.jnt.2019.08.010)

[Link to publication record in Explore Bristol Research](#)
PDF-document

This is the author accepted manuscript (AAM). The final published version (version of record) is available online via Elsevier at <https://www.sciencedirect.com/science/article/pii/S0022314X19302902#!>. Please refer to any applicable terms of use of the publisher.

University of Bristol - Explore Bristol Research

General rights

This document is made available in accordance with publisher policies. Please cite only the published version using the reference above. Full terms of use are available: <http://www.bristol.ac.uk/red/research-policy/pure/user-guides/ebr-terms/>

TEST VECTORS FOR RANKIN–SELBERG L -FUNCTIONS

ANDREW R. BOOKER, M. KRISHNAMURTHY, AND MIN LEE

ABSTRACT. We study the local zeta integrals attached to a pair of generic representations (π, τ) of $\mathrm{GL}_n \times \mathrm{GL}_m$, $n > m$, over a p -adic field. Through a process of unipotent averaging we produce a pair of corresponding Whittaker functions whose zeta integral is non-zero, and we express this integral in terms of the Langlands parameters of π and τ . In many cases, these Whittaker functions also serve as a test vector for the associated Rankin–Selberg (local) L -function.

1. INTRODUCTION

Let F be a non-archimedean local field with ring of integers \mathfrak{o} and residue field of cardinality q . For $m < n$, let π and τ be irreducible admissible representations of $\mathrm{GL}_n(F)$ and $\mathrm{GL}_m(F)$, respectively. We fix an additive character ψ of F with conductor \mathfrak{o} and assume that π and τ are generic relative to ψ .

Recall that the local *zeta integral* $\Psi(s; W, W')$ is defined by

$$(1.1) \quad \Psi(s; W, W') = \int_{U_m(F) \backslash \mathrm{GL}_m(F)} W \begin{pmatrix} h & \\ & I_{n-m} \end{pmatrix} W'(h) \|\det h\|^{s - \frac{n-m}{2}} dh,$$

where $W \in \mathcal{W}(\pi, \psi)$ and $W' \in \mathcal{W}(\tau, \psi^{-1})$ are Whittaker functions in the corresponding Whittaker spaces, and U_m is the group of unipotent matrices. It converges for $\Re(s) \gg 1$, and the collection of such zeta integrals spans a fractional ideal $\mathbb{C}[q^s, q^{-s}]L(s, \pi \boxtimes \tau)$ of the ring $\mathbb{C}[q^s, q^{-s}]$. We may choose the generator to satisfy $1/L(s, \pi \boxtimes \tau) \in \mathbb{C}[q^{-s}]$ and $\lim_{s \rightarrow \infty} L(s, \pi \boxtimes \tau) = 1$, and this gives the local Rankin–Selberg factor attached to the pair (π, τ) in [5, §2.7].

In particular, if we define a map

$$\mathcal{W}(\pi, \psi) \otimes \mathcal{W}(\tau, \psi^{-1}) \longrightarrow \mathbb{C}(q^{-s})$$

via

$$W \otimes W' \mapsto \Psi(s; W, W'),$$

then there is an element in $\mathcal{W}(\pi, \psi) \otimes \mathcal{W}(\tau, \psi^{-1})$ that maps to $L(s, \pi \boxtimes \tau)$. However, *a priori* this element need not be a pure tensor. In this paper, we produce a pure tensor $W \otimes W'$ for which the associated zeta integral is explicitly computable and non-zero. The precise result that we prove is the following.

Theorem 1.1. *Let $\{\alpha_i\}_{i=1}^n$ and $\{\gamma_j\}_{j=1}^m$ denote the Langlands parameters of π and τ , respectively, and let $L(s, \pi \times \tau)$ be the naive Rankin–Selberg L -factor defined by*

$$L(s, \pi \times \tau) = \prod_{i=1}^n \prod_{j=1}^m (1 - \alpha_i \gamma_j q^{-s})^{-1}.$$

A. R. B. was partially supported by EPSRC Grant EP/K034383/1. M. L. was supported by a Royal Society University Research Fellowship. M. K. was partially supported by International Programs and the Stanley–UI Foundation Support Organization. No data were created in the course of this study.

Then there is a pair $(W, W') \in \mathcal{W}(\pi, \psi) \times \mathcal{W}(\tau, \psi^{-1})$, described explicitly in §3, such that

$$\Psi(s; W, W') = L(s, \pi \times \tau).$$

When $\Psi(s; W, W') = L(s, \pi \boxtimes \tau)$, the pair (W, W') is called a *test vector* for (π, τ) . Hence the theorem produces a test vector whenever $L(s, \pi \times \tau) = L(s, \pi \boxtimes \tau)$ —for instance, if either π or τ is unramified or if $L(s, \pi \boxtimes \tau) = 1$. In general, one has $L(s, \pi \times \tau) = P(q^{-s})L(s, \pi \boxtimes \tau)$ for a non-zero polynomial $P \in \mathbb{C}[X]$ (see Lemma 2.1).

The overview of our method is as follows. Let ξ^0 (resp. φ^0) denote the “essential vector” in the space of π (resp. τ), and let $W_{\xi^0} \in \mathcal{W}(\pi, \psi)$ (resp. $W_{\varphi^0} \in \mathcal{W}(\tau, \psi^{-1})$) be the associated essential Whittaker functions, as described in detail in §2. When τ is unramified, it follows from [10, Corollary 3.3] that

$$(1.2) \quad L(s, \pi \boxtimes \tau) = \int_{U_m(F) \backslash GL_m(F)} W_{\xi^0} \begin{pmatrix} h & \\ & I_{n-m} \end{pmatrix} W_{\varphi^0}(h) \|\det h\|^{s - \frac{n-m}{2}} dh$$

for a suitable normalization of the measure on $U_m(F) \backslash GL_m(F)$. When $m = n - 1$, the above equality is part of the characterization of the essential vector in [4, 6]; the fact that it holds for any $m < n$ is the result of a concrete realization of essential functions in [10]. On the other hand, if τ is ramified then the local integral in (1.2) vanishes. Through a process of unipotent averaging (see (3.1) below), we modify W_{ξ^0} to obtain a Whittaker function $W \in \mathcal{W}(\pi, \psi)$ such that the resulting zeta integral $\Psi(s; W, W_{\varphi^0})$ equals $cL(s, \pi \times \tau)$ for a non-zero number $c \in \mathbb{C}$, depending on the conductor of τ , its central character ω_τ , and ψ . Setting $W' = c^{-1}W_{\varphi^0}$, we obtain the required pair (W, W') .

We mention some related results in the literature. First, if π and τ are discrete series representations, then the existence of a test vector (W, W') was shown in [9], but the Whittaker function W' there is taken to be in a larger space, namely the Whittaker space associated to the standard module of τ . Second, the so-called *local Birch lemma*, arising in the context of \mathfrak{p} -adic interpolation of special values of twisted Rankin–Selberg (global) L -functions, is also related. It concerns evaluation of a local integral in the special case that π is unramified and τ is the twist of an unramified representation by a character with non-trivial conductor; see [8, Proposition 3.1] and [7, Theorem 2.1]. The approach in [8] is similar to ours in that it also uses a process of unipotent averaging in order to modify the Whittaker function on the larger general linear group. We can of course apply Theorem 1.1 to their setup: Since $L(s, \pi \times \tau) = 1$ in this case, the pair (W, W_{φ^0}) described above has the property that $\Psi(s; W, W_{\varphi^0})$ is an explicit constant (independent of s).

Finally, note that one can obtain a global version of Theorem 1.1 by combining the test vectors at all (finite) places. In work in progress, we study the analogous question over an archimedean local field.

Acknowledgements. The second author (M. K.) would like to thank A. Raghuram for pointing him to [8].

2. PRELIMINARIES

Let \mathfrak{p} be the unique maximal ideal in \mathfrak{o} . We fix a generator ϖ of \mathfrak{p} with absolute value $\|\varpi\| = q^{-1}$. Let $v : F^\times \rightarrow \mathbb{Z}$ denote the valuation map, and extend it to fractional ideals in the usual way. For any $n > 1$, let $B_n = T_n U_n$ be the Borel subgroup of GL_n consisting of upper triangular matrices; let $P'_n \supset B_n$ be the standard parabolic subgroup of type

$(n-1, 1)$ with Levi decomposition $P'_n = M_n N_n$. Then $M_n \cong \mathrm{GL}_{n-1} \times \mathrm{GL}_1$ and

$$N_n = \left\{ \begin{pmatrix} I_{n-1} & * \\ & 1 \end{pmatrix} \right\}.$$

Also, we write Z_n to denote the center consisting of scalar matrices and $A_n \subset T_n$ to denote the subtorus consisting of diagonal matrices with lower-right entry 1.

If R is any F -algebra and H is any algebraic F -group, we write $H(R)$ to denote the corresponding group of R -points. Let $P_n(R) \subset P'_n(R)$ denote the mirabolic subgroup consisting of matrices whose last row is of the form $(0, \dots, 0, 1)$, i.e.,

$$P_n(R) = \left\{ \begin{pmatrix} h & y \\ & 1 \end{pmatrix} : h \in \mathrm{GL}_{n-1}(R), y \in R^{n-1} \right\} \cong \mathrm{GL}_{n-1}(R) \ltimes N_n(R).$$

The character

$$u \mapsto \psi \left(\sum_{i=1}^{n-1} u_{i,i+1} \right) \quad \text{for } u \in U_n(F)$$

defines a *generic character* of $U_n(F)$, and by abuse of notation we continue to denote this character by ψ . Further, for any algebraic subgroup $V \subseteq U_n$, ψ defines a character of $V(F)$ via restriction. In particular, we may consider the character $\psi|_{N_n(F)}$; its stabilizer in $M_n(F)$ is then $P_{n-1}(F)$, where we regard P_{n-1} as a subgroup of M_n via $h \mapsto \begin{pmatrix} h & \\ & 1 \end{pmatrix}$.

An irreducible representation (π, V_π) of $\mathrm{GL}_n(F)$ is said to be *generic* if

$$\mathrm{Hom}_{\mathrm{GL}_n(F)}(V_\pi, \mathrm{Ind}_{U_n(F)}^{\mathrm{GL}_n(F)} \psi) \neq 0.$$

By Frobenius reciprocity, this means that there is a non-zero linear form $\lambda : V_\pi \rightarrow \mathbb{C}$ satisfying $\lambda(\pi(u)v) = \psi(u)\lambda(v)$ for $v \in V_\pi, u \in U_n(F)$. It is known (see [3]) that for a generic π the space of such linear functionals, or equivalently the space $\mathrm{Hom}_{\mathrm{GL}_n(F)}(V_\pi, \mathrm{Ind}_{U_n(F)}^{\mathrm{GL}_n(F)} \psi)$, is of dimension 1. Let $\mathcal{W}(\pi, \psi)$ denote the Whittaker model of π , viz. the space of functions W_v on $\mathrm{GL}_n(F)$ defined by $W_v(g) = \lambda(\pi(g)v)$ for $v \in V_\pi$. Then $\mathcal{W}(\pi, \psi)$ is independent of the choice of λ , and for $u \in U_n(F), g \in \mathrm{GL}_n(F)$,

$$\begin{aligned} W_v(ug) &= \psi(u)W_v(g), \\ W_v(g) &= W_{\pi(g)v}(I_n). \end{aligned}$$

We will consider certain compact open subgroups of $\mathrm{GL}_n(F)$; namely, for any integer $f \geq 0$, set

$$\begin{aligned} K_1(\mathfrak{p}^f) &= \left\{ g \in \mathrm{GL}_n(\mathfrak{o}) : g \equiv \begin{pmatrix} & * \\ & \vdots \\ 0 \dots 0 & 1 \end{pmatrix} \pmod{\mathfrak{p}^f} \right\}, \\ K_0(\mathfrak{p}^f) &= \left\{ g \in \mathrm{GL}_n(\mathfrak{o}) : g \equiv \begin{pmatrix} & * \\ & \vdots \\ 0 \dots 0 & * \end{pmatrix} \pmod{\mathfrak{p}^f} \right\}, \end{aligned}$$

so that $K_1(\mathfrak{p}^f)$ is a normal subgroup of $K_0(\mathfrak{p}^f)$, with quotient $K_0(\mathfrak{p}^f)/K_1(\mathfrak{p}^f) \cong (\mathfrak{o}/\mathfrak{p}^f)^\times$.

Next we introduce our choice of measures. For $n \geq 1$ we normalize the Haar measure on $\mathrm{GL}_n(F)$ and $\mathrm{GL}_n(\mathfrak{o})$ so that $\mathrm{vol}(\mathrm{GL}_n(\mathfrak{o})) = 1$, and we fix the Haar measure on $U_n(F)$ for which $\mathrm{vol}(U_n(F) \cap \mathrm{GL}_n(\mathfrak{o})) = 1$. From these, we obtain a right-invariant measure on $U_n(F) \backslash \mathrm{GL}_n(F)$. We may make this explicit using the Iwasawa decomposition. For instance, let dx be the Haar measure on F such that \mathfrak{o} has unit volume, and let $d^\times x$ be the multiplicative measure on F^\times such that $\mathrm{vol}(\mathfrak{o}^\times) = 1$, i.e., $d^\times x = \frac{q}{q-1} \frac{dx}{\|x\|}$. Let dz and da be the corresponding measures on the center $Z_n(F) \cong F^\times$ and the subtorus $A_n(F) \cong$

$(F^\times)^{n-1}$, respectively. We fix the isomorphism $(F^\times)^{n-1} \cong A_n(F)$ via $(a_1, \dots, a_{n-1}) \mapsto a = t(a_1, \dots, a_{n-1})$, where

$$(2.1) \quad t(a_1, \dots, a_{n-1}) = \begin{pmatrix} a_1 a_2 \dots a_{n-1} & & & & \\ & a_1 a_2 \dots a_{n-2} & & & \\ & & \ddots & & \\ & & & a_1 & \\ & & & & 1 \end{pmatrix}.$$

Then $da = d^\times a_1 d^\times a_2 \dots d^\times a_{n-1}$. If $f \in C_c^\infty(\mathrm{GL}_n(F))$ is $\mathrm{U}_n(F)$ -invariant on the left, we then have the integration formula

$$(2.2) \quad \int_{\mathrm{U}_n(F) \backslash \mathrm{GL}_n(F)} f(g) dg = \int_{\mathrm{Z}_n(F) \times A_n(F) \times \mathrm{GL}_n(\mathfrak{o})} f(zak) \delta_{\mathrm{B}_n}(a)^{-1} dz da dk,$$

where δ_{B_n} is the modulus character, defined so that

$$(2.3) \quad \delta_{\mathrm{B}_n}(a) = \prod_{i=1}^{n-1} \|a_i\|^{i(n-i)}.$$

Next we review the notion of *conductor* and the theory of the *essential vector* associated to an irreducible, admissible, generic representation π . According to [4] (see also [6]), there is a unique positive integer $m(\pi)$ such that the space of $K_1(\mathfrak{p}^{m(\pi)})$ -fixed vectors is 1-dimensional. Further, as alluded to in the introduction, by loc. cit. there is a unique vector ξ^0 in this space, called the essential vector, with the associated essential function $W_{\xi^0} \in \mathcal{W}(\pi, \psi)$ satisfying the condition $W_{\xi^0} \begin{pmatrix} g & \\ & 1 \end{pmatrix} = W_{\xi^0} \begin{pmatrix} g & \\ & 1 \end{pmatrix}$ for all $h \in \mathrm{GL}_{n-1}(\mathfrak{o})$ and $g \in \mathrm{GL}_{n-1}(F)$. Since U_n acts via ψ on the left, it follows that

$$(2.4) \quad W_{\xi^0}(t(a_1, \dots, a_{n-1})) \neq 0 \implies a_1, \dots, a_{n-1} \in \mathfrak{o}.$$

If π is unramified, let $W_\pi^{0, \psi} \in \mathcal{W}(\pi, \psi)$ denote the normalized spherical function [6, p. 2]. If $m(\pi) = 0$ then by uniqueness of essential functions, one has the equality $W_{\xi^0} = W_\pi^{0, \psi}$. The integral ideal $\mathfrak{p}^{m(\pi)}$ is called the *conductor* of π . In passing, we mention that the integer $m(\pi)$ can also be characterized as the degree of the monomial in the local ϵ -factor $\epsilon(s, \pi, \psi)$ [4], i.e. so that

$$\epsilon(s, \pi, \psi) = \epsilon(\pi, \psi) q^{m(\pi)(\frac{1}{2}-s)}$$

for some $\epsilon(\pi, \psi) \in \mathbb{C}^\times$.

A crucial property of the conductor is that $K_0(\mathfrak{p}^{m(\pi)})$ acts on the space of $K_1(\mathfrak{p}^{m(\pi)})$ -fixed vectors via the central character ω_π (cf. [2, Section 8]). Precisely, for $g = (g_{i,j}) \in K_0(\mathfrak{p}^{m(\pi)})$, define

$$\chi_\pi(g) = \begin{cases} 1 & \text{if } m(\pi) = 0, \\ \omega_\pi(g_{n,n}) & \text{if } m(\pi) > 0. \end{cases}$$

It is shown in loc. cit. that χ_π is a character of $K_0(\mathfrak{p}^{m(\pi)})$ trivial on $K_1(\mathfrak{p}^{m(\pi)})$, and

$$\pi(g)\xi^0 = \chi_\pi(g)\xi^0 \quad \text{for all } g \in K_0(\mathfrak{p}^{m(\pi)}).$$

We end this section by recalling the definition of conductor of a multiplicative character χ of F^\times . If χ is trivial on \mathfrak{o}^\times then the conductor of χ is \mathfrak{o} ; otherwise, the conductor is \mathfrak{p}^n , where $n \geq 1$ is the least integer such that χ is trivial on $1 + \mathfrak{p}^n$.

2.1. Rankin–Selberg L -functions. In this subsection alone we drop the assumption that $m < n$ and allow (m, n) to be an arbitrary pair of positive integers. For π and τ irreducible, admissible, generic representations of $\mathrm{GL}_n(F)$ and $\mathrm{GL}_m(F)$, respectively, let $L(s, \pi \boxtimes \tau)$ be as defined in [5]. When $m < n$, $L(s, \pi \boxtimes \tau)$ is defined as in the introduction. For $m > n$, one defines $L(s, \pi \boxtimes \tau) = L(s, \tau \boxtimes \pi)$. For $m = n$, the defining local integrals are different and involve a Schwartz function on F^n ; see loc. cit.

Next we elaborate on the definition of the naive Rankin–Selberg L -factor, $L(s, \pi \times \tau)$, introduced in Theorem 1.1. By definition, the L -function $L(s, \pi)$ is of the form $P_\pi(q^{-s})^{-1}$, where $P_\pi \in \mathbb{C}[X]$ has degree at most n and satisfies $P_\pi(0) = 1$. We may then find n complex numbers $\{\alpha_i\}_{i=1}^n$ (allowing some of them to be zero) satisfying

$$L(s, \pi) = \prod_{i=1}^n (1 - \alpha_i q^{-s})^{-1}.$$

We call the set $\{\alpha_i\}$ the *Langlands parameters* of π ; if π is spherical, they agree with the usual Satake parameters. Let $\{\gamma_j\}_{j=1}^m$ be the Langlands parameters of τ , and set

$$(2.5) \quad L(s, \pi \times \tau) = \prod_{i=1}^n \prod_{j=1}^m (1 - \alpha_i \gamma_j q^{-s})^{-1}.$$

$L(s, \pi \times \tau) = L(s, \pi \boxtimes \tau)$ if π or τ is spherical.

In the following lemma we describe the connection between $L(s, \pi \times \tau)$ and $L(s, \pi \boxtimes \tau)$. To that end, we first recall the classification of irreducible admissible representations of $\mathrm{GL}_n(F)$. Let \mathcal{A}_n denote the set of equivalence classes of such representations, and put $\mathcal{A} = \bigcup \mathcal{A}_n$. The essentially square-integrable representations of $\mathrm{GL}_n(F)$ have been classified by Bernstein and Zelevinsky, and they are as follows. If σ is an essentially square-integrable representation of $\mathrm{GL}_n(F)$, then there is a divisor $a \mid n$ and a supercuspidal representation η of $\mathrm{GL}_a(F)$ such that if $b = \frac{n}{a}$ and Q is the standard (upper) parabolic subgroup of $\mathrm{GL}_n(F)$ of type (a, \dots, a) , then σ can be realized as the unique quotient of the (normalized) induced representation

$$\mathrm{Ind}_Q^{\mathrm{GL}_n(F)}(\eta, \eta \| \cdot \|, \dots, \eta \| \cdot \|^{b-1}).$$

The integer a and the class of η are uniquely determined by σ . In short, σ is parametrized by b and η , and we denote this by $\sigma = \sigma_b(\eta)$; further, σ is square-integrable (also called “discrete series”) if and only if the representation $\eta \| \cdot \|^{-\frac{b-1}{2}}$ of $\mathrm{GL}_a(F)$ is unitary.

Now, let P be an upper parabolic subgroup of $\mathrm{GL}_n(F)$ of type (n_1, \dots, n_r) . For each $i = 1, \dots, r$, let τ_i^0 be a discrete series representation of $\mathrm{GL}_{n_i}(F)$. Let (s_1, \dots, s_r) be a sequence of real numbers satisfying $s_1 \geq \dots \geq s_r$, and put $\tau_i = \tau_i^0 \otimes \| \cdot \|^{s_i}$ (an essentially square-integrable representation). Then the induced representation

$$\xi = \mathrm{Ind}_P^{\mathrm{GL}_n(F)}(\tau_1 \otimes \dots \otimes \tau_r)$$

is said to be a representation of $\mathrm{GL}_n(F)$ of Langlands type. If $\tau \in \mathcal{A}_n$, then it is well known that it is uniquely representable as the quotient of an induced representation of Langlands type. We write $\tau = \tau_1 \boxplus \dots \boxplus \tau_r$ to denote this realization of τ . Thus one obtains a sum operation on the set \mathcal{A} [5, §9.5]. It follows easily from the definition that $L(s, \pi \times \tau)$ is bi-additive, i.e.

$$\begin{aligned} L(s, \pi \times (\tau \boxplus \tau')) &= L(s, \pi \times \tau) L(s, \pi \times \tau') \\ L(s, (\pi \boxplus \pi') \times \tau) &= L(s, \pi \times \tau) L(s, \pi' \times \tau) \end{aligned}$$

for all $\pi, \pi', \tau, \tau' \in \mathcal{A}$. The local factor $L(s, \pi \boxtimes \tau)$ is also bi-additive in the above sense, by [5, §9.5, Theorem].

Lemma 2.1. *Let m and n be positive integers, and consider $\pi \in \mathcal{A}_n$, $\tau \in \mathcal{A}_m$. Then*

$$L(s, \pi \times \tau) = P(q^{-s})L(s, \pi \boxtimes \tau)$$

for a polynomial $P \in \mathbb{C}[X]$ (depending on π and τ) satisfying $P(0) = 1$.

Proof. Since π and τ are sums of essentially square-integrable representations and $L(s, \pi \times \tau)$ and $L(s, \pi \boxtimes \tau)$ are both additive with respect to \boxplus , it suffices to prove the lemma for a pair (π, τ) of essentially square-integrable representations. In particular, assume $\pi = \sigma_b(\eta)$ as above.

We proceed by induction on m . If $m = 1$ then $\tau = \chi$ is a quasi-character of F^\times and $L(s, \pi \boxtimes \tau) = L(s, \pi \otimes \chi)$, where $\pi \otimes \chi$ is the representation of $\mathrm{GL}_n(F)$ defined by $g \mapsto \pi(g)\chi(\det g)$. If χ is unramified, then

$$L(s, \pi \otimes \chi) = L(s, \pi \times \chi),$$

and consequently $P = 1$. On the other hand, if χ is ramified then $L(s, \pi \times \chi) = 1$, and the assertion follows since $L(s, \pi \otimes \chi)^{-1}$ is a polynomial in q^{-s} .

We now assume $m > 1$ and τ is an essentially square-integrable representation of $\mathrm{GL}_m(F)$, say $\tau = \sigma_{b'}(\eta')$, where $\eta' \in \mathcal{A}_{a'}$ is supercuspidal and $a'b' = m$. Then the standard L -factor $L(s, \tau)$ is given by $L(s, \tau) = L(s + b' - 1, \eta')$ [5]. Therefore, $L(s, \tau) = 1$ unless $a' = 1$ and $\eta' = \chi$ is an unramified quasi-character of F^\times . On the other hand, if $L(s, \tau) = 1$, then $L(s, \pi \times \tau) = 1$ and the assertion of the lemma follows. Hence we may assume $\tau = \sigma_m(\chi)$ for an unramified quasi-character χ of F^\times , in which case

$$(2.6) \quad \begin{aligned} L(s, \pi \times \tau) &= L(s, \pi \otimes \chi \parallel \cdot \parallel^{m-1}) = L(s, \sigma_b(\eta) \otimes \chi \parallel \cdot \parallel^{m-1}) \\ &= L(s + m - 1 + b - 1, \eta \otimes \chi). \end{aligned}$$

On the other hand, it follows from [5, §8.2, Theorem] that

$$(2.7) \quad L(s, \pi \boxtimes \tau) = \begin{cases} \prod_{j=0}^{m-1} L(s + j + b - 1, \eta \otimes \chi) & \text{if } m \leq n, \\ \prod_{i=0}^{b-1} L(s + m - 1 + i, \eta \otimes \chi) & \text{if } m > n. \end{cases}$$

From (2.6) and (2.7), one sees that the ratio $\frac{L(s, \pi \times \tau)}{L(s, \pi \boxtimes \tau)}$ is a polynomial in q^{-s} , thus proving the lemma. \square

Corollary 2.2. *If $L(s, \pi \boxtimes \tau) = 1$ then either $L(s, \pi) = 1$ or $L(s, \tau) = 1$.*

Proof. If $L(s, \pi \boxtimes \tau) = 1$ then Lemma 2.1 implies that $L(s, \pi \times \tau)$ is a polynomial in q^{-s} , and hence must be 1. This in turn implies the conclusion. \square

3. THE MAIN CALCULATION

Recall that ξ^0 and φ^0 are the essential vectors of π and τ , respectively. Here we construct a pair $(W, W') \in \mathcal{W}(\pi, \psi) \times \mathcal{W}(\tau, \psi^{-1})$ as in Theorem 1.1. Let \mathfrak{n} , \mathfrak{q} and \mathfrak{c} denote the conductors of π , τ and ω_τ , respectively. If τ is an unramified representation of $\mathrm{GL}_m(F)$, then by (1.2) we have

$$\Psi(s; W_{\xi^0}, W_{\varphi^0}) = L(s, \pi \boxtimes \tau) = L(s, \pi \times \tau).$$

Thus, in this case we can take $(W, W') = (W_{\xi^0}, W_{\varphi^0})$.

Let us assume from now on that τ is ramified, meaning $v(\mathfrak{q}) > 0$. Since $\mathfrak{c} \supseteq \mathfrak{q}$, we have $v(\mathfrak{c}) \leq v(\mathfrak{q})$. Consider $\beta = (\beta_1, \dots, \beta_m) \in F^m$, with $\beta_i \in \mathfrak{q}^{-1}$ for $i = 1, \dots, m$, and

let $u(\beta)$ denote the $n \times n$ matrix with 1s on the diagonal and β^t embedded above the diagonal in the $(m+1)^{\text{st}}$ column. Let ξ_β^0 denote the vector $\xi_\beta^0 = \pi(u(\beta))\xi^0$, and define

$$(3.1) \quad \bar{\xi} = \frac{1}{[\mathfrak{o} : \mathfrak{q}]^{m-1}} \sum_{(\beta_1, \dots, \beta_{m-1}) \in (\mathfrak{q}^{-1}/\mathfrak{o})^{m-1}} \xi_{(\beta_1, \dots, \beta_{m-1}, \varpi^{-v(\mathfrak{c})})}^0.$$

(When $m = 1$ we understand there to be one summand, so that $\bar{\xi} = \xi_{(\varpi^{-v(\mathfrak{c})})}^0$.) We will now calculate $\Psi(s; W_{\bar{\xi}}, W_{\varphi^0})$, which by linearity equals

$$\frac{1}{[\mathfrak{o} : \mathfrak{q}]^{m-1}} \sum_{(\beta_1, \dots, \beta_{m-1}) \in (\mathfrak{q}^{-1}/\mathfrak{o})^{m-1}} \Psi(s; W_{\xi_{(\beta_1, \dots, \beta_{m-1}, \varpi^{-v(\mathfrak{c})})}^0}, W_{\varphi^0}).$$

Put $K = \text{GL}_m(\mathfrak{o})$. By (2.2), for fixed $\beta = (\beta_1, \dots, \beta_m)$, we have

$$\begin{aligned} \Psi(s; W_{\xi_\beta^0}, W_{\varphi^0}) &= \int_{Z_m(F) \times A_m(F) \times K} \omega_\tau(z) W_{\xi^0} \left(\begin{pmatrix} z a k & \\ & I_{n-m} \end{pmatrix} u(\beta) \right) W_{\varphi^0}(a k) \\ &\quad \cdot \delta_{B_m}(a)^{-1} \|\det(z a)\|^{s - \frac{n-m}{2}} dz da dk \\ &= \int_{F^\times \times A_m(F) \times K} \omega_\tau(z) \left(\sum_{j=1}^m \psi(z k_j \beta_j) \right) W_{\xi^0} \begin{pmatrix} z a & \\ & I_{n-m} \end{pmatrix} W_{\varphi^0}(a k) \\ &\quad \cdot \delta_{B_m}(a)^{-1} \|z^m \det a\|^{s - \frac{n-m}{2}} d^\times z da dk, \end{aligned}$$

where (k_1, \dots, k_m) is the bottom row of the matrix k . Here we have used the fact that the function $h \mapsto W_{\xi^0} \begin{pmatrix} h & \\ & I_{n-m} \end{pmatrix}$, $h \in \text{GL}_m(F)$, is right K -invariant. Now, performing the average over $\beta_j \in \mathfrak{q}^{-1}/\mathfrak{o}$ for each $j < m$, we see that $\Psi(s; W_{\bar{\xi}}, W_{\varphi^0})$ equals

$$\begin{aligned} &\int_{\left\{ \begin{array}{l} F^\times \times A_m(F) \times K \\ z k_j \in \mathfrak{q} \ \forall j < m \end{array} \right\}} \omega_\tau(z) \psi(z k_m \varpi^{-v(\mathfrak{c})}) W_{\xi^0} \begin{pmatrix} z a & \\ & I_{n-m} \end{pmatrix} W_{\varphi^0}(a k) \\ &\quad \cdot \delta_{B_m}(a)^{-1} \|z^m \det a\|^{s - \frac{n-m}{2}} d^\times z da dk. \end{aligned}$$

By (2.4) we have $W_{\xi^0}(t(a_1, \dots, a_{n-1})) = 0$ unless $a_1, \dots, a_{n-1} \in \mathfrak{o}$. In view of (2.1), it follows that the integrand vanishes unless z is integral.

Note that

$$z k_j \in \mathfrak{q} \ \forall j < m \iff k \in K_0(z^{-1} \mathfrak{q} \cap \mathfrak{o}).$$

For $r \in \mathbb{Z}_{\geq 0}$, put

$$\begin{aligned} \Psi_r &= \int_{A_m(F) \times K_0(\mathfrak{q} \mathfrak{p}^{-r} \cap \mathfrak{o})} \mathcal{G}(\omega_\tau, \psi, \varpi^{r-v(\mathfrak{c})} k_m) W_{\xi^0} \begin{pmatrix} \varpi^r a & \\ & I_{n-m} \end{pmatrix} W_{\varphi^0}(a k) \\ &\quad \cdot \delta_{B_m}(a)^{-1} \|\det a\|^{s - \frac{n-m}{2}} da dk, \end{aligned}$$

where $\mathcal{G}(\omega_\tau, \psi, y)$ denotes the Gauss sum

$$\mathcal{G}(\omega_\tau, \psi, y) = \int_{\mathfrak{o}^\times} \omega_\tau(z) \psi(y z) d^\times z.$$

Then we have

$$\Psi(s; W_{\bar{\xi}}, W_{\varphi^0}) = \sum_{r \geq 0} \omega_\tau(\varpi^r) q^{-rm(s - \frac{n-m}{2})} \Psi_r.$$

Suppose that ω_τ is ramified, so that $v(\mathfrak{c}) > 0$. Then $\mathcal{G}(\omega_\tau, \psi, \varpi^{r-v(\mathfrak{c})}k_m)$ vanishes unless $v(\varpi^{r-v(\mathfrak{c})}k_m) = -v(\mathfrak{c})$, which implies $v(k_m) = -r$. Since k_m is integral, it follows that $r = 0$ is the only contributing term to $\Psi(s; W_{\bar{\xi}}, W_{\varphi^0})$, so that

$$\begin{aligned} \Psi(s; W_{\bar{\xi}}, W_{\varphi^0}) &= \int_{A_m(F) \times K_0(\mathfrak{q})} \mathcal{G}(\omega_\tau, \psi, \varpi^{-v(\mathfrak{c})}k_m) W_{\xi^0} \begin{pmatrix} a & \\ & I_{n-m} \end{pmatrix} W_{\varphi^0}(ak) \\ &\quad \cdot \delta_{B_m}(a)^{-1} \|\det a\|^{s-\frac{n-m}{2}} da dk. \end{aligned}$$

Moreover, since $k_m \in \mathfrak{o}^\times$, we have $\mathcal{G}(\omega_\tau, \psi, \varpi^{-v(\mathfrak{c})}k_m) = \omega_\tau(k_m)^{-1} \mathcal{G}(\omega_\tau, \psi, \varpi^{-v(\mathfrak{c})})$, and thus

$$(3.2) \quad \Psi(s; W_{\bar{\xi}}, W_{\varphi^0}) = c \int_{A_m(F)} W_{\xi^0} \begin{pmatrix} a & \\ & I_{n-m} \end{pmatrix} W_{\varphi^0}(a) \delta_{B_m}(a)^{-1} \|\det a\|^{s-\frac{n-m}{2}} da,$$

where

$$c = \frac{\mathcal{G}(\omega_\tau, \psi, \varpi^{-v(\mathfrak{c})})}{[\mathrm{GL}_m(\mathfrak{o}) : K_0(\mathfrak{q})]} \neq 0.$$

Suppose now that ω_τ is unramified, so that $\mathfrak{c} = \mathfrak{o}$ and $m > 1$. Then $\mathcal{G}(\omega_\tau, \psi, \varpi^{r-v(\mathfrak{c})}k_m) = 1$ for all r . If $r > 0$ then $K_0(\mathfrak{q}\mathfrak{p}^{-r} \cap \mathfrak{o}) \subsetneq K_0(\mathfrak{q})$; since the conductor of τ is \mathfrak{q} , it follows from [4, Theorem 5.1] that

$$\int_{K_0(\mathfrak{q}\mathfrak{p}^{-r} \cap \mathfrak{o})} W_{\varphi^0}(ak) dk = 0,$$

which in turn implies that $\Psi_r = 0$. Hence only the $r = 0$ term contributes, and again we arrive at (3.2).

It remains only to identify the integral over $A_m(F)$.

Lemma 3.1. *When τ is ramified, we have*

$$\int_{A_m(F)} W_{\xi^0} \begin{pmatrix} a & \\ & I_{n-m} \end{pmatrix} W_{\varphi^0}(a) \delta_{B_m}(a)^{-1} \|\det a\|^{s-\frac{n-m}{2}} da = L(s, \pi \times \tau).$$

Proof. Let $\alpha = (\alpha_1, \dots, \alpha_n)$ and $(\gamma_1, \dots, \gamma_m)$ denote the Langlands parameters of π and τ , respectively. Since τ is ramified, we may take $\gamma_m = 0$. We set $\gamma_{m+1} = \dots = \gamma_n = 0$ and write $\gamma = (\gamma_1, \dots, \gamma_n)$.

Writing $a = t(a_1, \dots, a_{m-1})$ as in (2.1), by (2.4) we see that $W_{\varphi^0}(a)$ vanishes unless each a_i is integral. Setting

$$\lambda_i = \sum_{1 \leq j \leq m-i} v(a_j) \quad \text{for } i = 1, \dots, n,$$

the integral in question may be written as

$$\sum_{\substack{\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}_{\geq 0}^n \\ \lambda_1 \geq \dots \geq \lambda_{m-1} \\ \lambda_m = \dots = \lambda_n = 0}} W_{\xi^0}(d_n(\lambda)) W_{\varphi^0}(d_m(\lambda)) \delta_{B_m}(d_m(\lambda))^{-1} \|\det d_m(\lambda)\|^{s-\frac{n-m}{2}},$$

where

$$d_n(\lambda) = \mathrm{diag}(\varpi^{\lambda_1}, \dots, \varpi^{\lambda_n}) \quad \text{and} \quad d_m(\lambda) = \mathrm{diag}(\varpi^{\lambda_1}, \dots, \varpi^{\lambda_m}).$$

On the other hand, by [11, Theorem 4.1], we have

$$W_{\xi^0}(d_n(\lambda)) = \delta_{B_n}(d_n(\lambda))^{\frac{1}{2}} s_\lambda(\alpha) \quad \text{and} \quad W_{\varphi^0}(d_m(\lambda)) = \delta_{B_m}(d_m(\lambda))^{\frac{1}{2}} s_\lambda(\gamma),$$

where s_λ denotes the Schur polynomial

$$s_\lambda(X_1, \dots, X_n) = \frac{\det(X_j^{\lambda_i + n - i})_{1 \leq i, j \leq n}}{\prod_{1 \leq i < j \leq n} (X_i - X_j)}$$

(see [1, Theorem 38.1]). Thus the integral becomes

$$\begin{aligned} & \sum_{\substack{\lambda=(\lambda_1, \dots, \lambda_n) \in \mathbb{Z}_{\geq 0}^n \\ \lambda_1 \geq \dots \geq \lambda_{m-1} \\ \lambda_m = \dots = \lambda_n = 0}} s_\lambda(\alpha) s_\lambda(\gamma) \left(\frac{\delta_{B_n}(d_n(\lambda))}{\delta_{B_m}(d_m(\lambda))} \right)^{\frac{1}{2}} \|\det d_m(\lambda)\|^{s - \frac{n-m}{2}} \\ &= \sum_{\substack{\lambda=(\lambda_1, \dots, \lambda_n) \in \mathbb{Z}_{\geq 0}^n \\ \lambda_1 \geq \dots \geq \lambda_n}} s_\lambda(\alpha) s_\lambda(\gamma) \|\det d_m(\lambda)\|^s, \end{aligned}$$

where the last line follows from (2.3) and the fact that $\gamma_m = \dots = \gamma_n = 0$.

By the Cauchy identity [1, Theorem 43.3], for sufficiently small $x \in \mathbb{C}$, we have

$$\prod_{i=1}^n \prod_{j=1}^n (1 - x \alpha_i \gamma_j)^{-1} = \sum_{\substack{\lambda=(\lambda_1, \dots, \lambda_n) \in \mathbb{Z}_{\geq 0}^n \\ \lambda_1 \geq \dots \geq \lambda_n}} s_\lambda(x\alpha) s_\lambda(\gamma) = \sum_{\substack{\lambda=(\lambda_1, \dots, \lambda_n) \in \mathbb{Z}_{\geq 0}^n \\ \lambda_1 \geq \dots \geq \lambda_n}} s_\lambda(\alpha) s_\lambda(\gamma) x^{\lambda_1 + \dots + \lambda_n}.$$

Therefore,

$$\sum_{\substack{\lambda=(\lambda_1, \dots, \lambda_n) \in \mathbb{Z}_{\geq 0}^n \\ \lambda_1 \geq \dots \geq \lambda_n}} s_\lambda(\alpha) s_\lambda(\gamma) \|\det d_m(\lambda)\|^s = \prod_{i=1}^n \prod_{j=1}^n (1 - \alpha_i \gamma_j q^{-s})^{-1} = L(s, \pi \times \tau).$$

□

Finally, we take $W = W_{\bar{\xi}}$ and $W' = c^{-1}W_{\varphi^0}$ to conclude the proof of Theorem 1.1.

REFERENCES

1. Daniel Bump, *Lie groups*, second ed., Graduate Texts in Mathematics, vol. 225, Springer, New York, 2013. MR 3136522
2. J. W. Cogdell and I. I. Piatetski-Shapiro, *Converse theorems for GL_n* , Inst. Hautes Études Sci. Publ. Math. (1994), no. 79, 157–214. MR 1307299
3. I. M. Gelfand and D. A. Kajdan, *Representations of the group $GL(n, K)$ where K is a local field*, (1975), 95–118. MR 0404534
4. H. Jacquet, I. I. Piatetski-Shapiro, and J. Shalika, *Conducteur des représentations du groupe linéaire*, Math. Ann. **256** (1981), no. 2, 199–214. MR 620708
5. H. Jacquet, I. I. Piatetskii-Shapiro, and J. A. Shalika, *Rankin-Selberg convolutions*, Amer. J. Math. **105** (1983), no. 2, 367–464. MR 701565
6. Hervé Jacquet, *A correction to conducteur des représentations du groupe linéaire [mr620708]*, Pacific J. Math. **260** (2012), no. 2, 515–525. MR 3001803
7. Fabian Januszewski, *Modular symbols for reductive groups and p -adic Rankin-Selberg convolutions over number fields*, J. Reine Angew. Math. **653** (2011), 1–45. MR 2794624
8. D. Kazhdan, B. Mazur, and C.-G. Schmidt, *Relative modular symbols and Rankin-Selberg convolutions*, J. Reine Angew. Math. **519** (2000), 97–141. MR 1739728
9. R. Kurinczuk and N. Matringe, *Extension of Whittaker functions and test vectors*, Res. Number Theory **4** (2018), no. 3, Art. 31, 18. MR 3831387
10. Nadir Matringe, *Essential Whittaker functions for $GL(n)$* , Doc. Math. **18** (2013), 1191–1214. MR 3138844
11. Michitaka Miyauchi, *Whittaker functions associated to newforms for $GL(n)$ over p -adic fields*, J. Math. Soc. Japan **66** (2014), no. 1, 17–24. MR 3161390

E-mail address: `andrew.booker@bristol.ac.uk`

E-mail address: `muthu-krishnamurthy@uiowa.edu`

E-mail address: `min.lee@bristol.ac.uk`

SCHOOL OF MATHEMATICS, UNIVERSITY OF BRISTOL, UNIVERSITY WALK, BRISTOL, BS8 1TW,
UNITED KINGDOM

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF IOWA, 14 MACLEAN HALL, IOWA CITY, IA
52242-1419, USA